Seminar “Statistics for structures”

A graphical perspective on Gauss-Markov process priors

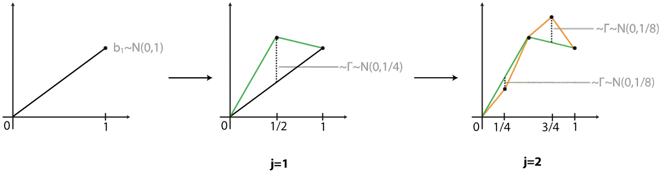
Moritz Schauer
University of Amsterdam

Outline

- Midpoint displacement construction of a Brownian motion
- Corresponding Gaussian Markov random field
- Chordal graphs
- Sparse Cholesky decomposition
- Connection to inference of diffusion processes

Mid-point displacement

Lévy-Ciesielski construction of a Brownian motion $(W_t)_{t\in[0,1]}$



[1]

Faber-Schauder basis

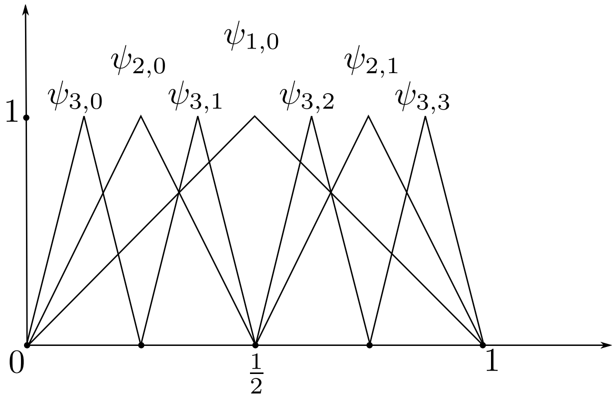


Figure: Elements $\psi_{l,k}$, $1 \leq l \leq 3$ of the hierarchical (Faber-) Schauder basis

Schauder basis functions

A location and scale family based on the "hat" function
$\bigwedge(x) = (2x)\mathbf{1}_{[0,\frac{1}{2})} + 2(x-1)\mathbf{1}_{[\frac{1}{2},1]}$

$$\psi_{j,k}(x) = \bigwedge(2^{j-1}x - k), \quad j \geq 1, \quad k = 0, \ldots, 2^{j-1} - 1$$

Mid-point displacement II

Start with Brownian motion bridge $(W_t)_{t\in[0,1]}$

$$W^J = \sum_{j=1}^{J} \sum_{k=0}^{2^{j-1}-1} Z_{j,k}\psi_{j,k}$$

W^J – truncated Faber–Schauder expansion

$$Z^J = \text{vec}\left(Z_{j,k}, j \le J, 0 \le k < 2^{j-1}\right)$$

Z^J – independent zero mean Gaussian random variables

$$Z_{j,k} = W_{2^{-j}(2k+1)} - \frac{1}{2}(W_{2^{-j+1}k} + W_{2^{-j+1}(k+1)})$$

Mid-point displacement II

Start with mean zero Gauss–Markov process $(W_t)_{t\in[0,1]}$

$$W^J = \sum_{j=1}^{J} \sum_{k=0}^{2^{j-1}-1} Z_{j,k}\psi_{j,k}$$

W^J – truncated Faber–Schauder expansion

$$Z^J = \mathrm{vec}\left(Z_{j,k}, j \le J, 0 \le k < 2^{j-1}\right)$$

Z^J – mean zero Gaussian vector with precision matrix Γ

$$Z_{j,k} = W_{2^{-j}(2k+1)} - \frac{1}{2}(W_{2^{-j+1}k} + W_{2^{-j+1}(k+1)})$$

Markov property

Write $\iota := (j,k)$, $\iota' = (j',k')$
In general

$$\Gamma_{\iota,\iota'} = 0 \quad \text{if} \quad Z_\iota \perp\!\!\!\perp Z_{\iota'} \mid Z_{\{\iota,\iota'\}^C}$$

By the Markov property

$$\Gamma_{\iota,\iota'} = 0 \quad \text{if} \quad \psi_\iota \cdot \psi_{\iota'} \equiv 0$$

Gaussian Markov random field

A Gaussian vector $(Z_1, \ldots, Z_n)$ together with the graph $\mathcal{G}(\{1, \ldots, n\}, \mathcal{E})$ where

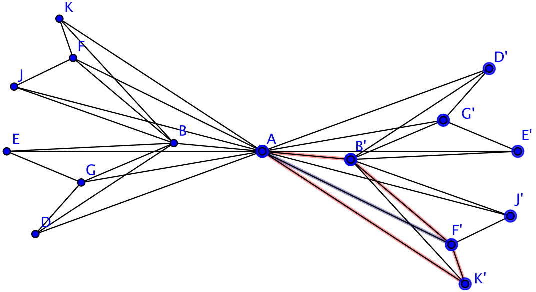
$$\text{no edge in } \mathcal{E} \text{ between } \iota \text{ and } \iota' \quad \text{if} \quad Z_\iota \perp\!\!\!\perp Z_{\iota'} \mid Z_{\{\iota, \iota'\}^C}$$

Chordal graph / Triangulated graph

"A *chordal graph* is a graph in which all cycles of four or more vertices have a *chord*, which is an edge that is not part of the cycle but connects two vertices of the cycle."

Interval graph

The open supports of $\psi_{j,k}$ form an *interval graph* on pairs (j,k). Interval graphs are chordal graphs.



In red a cycle of four vertices with a blue chord[1]

[1] An interval graph is the intersection graph of a family of intervals on the real line. Interval graphs are chordal graphs.

Sampling from the prior

- Sample J
- Compute factorization $SS' = \Gamma^J$
- Solve by backsubstitution

$$L'Z = \mathrm{WN}$$

with WN – standard white noise

Hence: How to find sparse factors?

Perfect elimination ordering

"A *perfect elimination ordering* in a graph is an ordering of the vertices of the graph such that, for each vertex v, v and the neighbors of v that occur after v in the order form a *clique*."
Example:

$$(3,0)\ (3,1)\ (3,2)\ (3,4)\ (2,0)\ (2,1)\ (1,0)$$

Ordering the columns and rows of Γ according to the perfect elimination ordering of the chordal graph:
$\tilde{S}$ is the sparse Cholesky factor of $\tilde{\Gamma}$

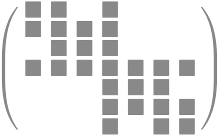
$$\tilde{\Gamma} = \begin{pmatrix} \blacksquare & & & & \blacksquare & & \blacksquare \\ & \blacksquare & & & \blacksquare & & \blacksquare \\ & & \blacksquare & & & \blacksquare & \blacksquare \\ & & & \blacksquare & & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & & & \blacksquare & & \blacksquare \\ & & \blacksquare & \blacksquare & & \blacksquare & \blacksquare \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{pmatrix} \quad \tilde{S} = \begin{pmatrix} \blacksquare & & & & & & \\ & \blacksquare & & & & & \\ & & \blacksquare & & & & \\ & & & \blacksquare & & & \\ \blacksquare & \blacksquare & & & \blacksquare & & \\ & & \blacksquare & \blacksquare & & \blacksquare & \\ \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare & \blacksquare \end{pmatrix}$$

Cholesky decomposition has no fill in!

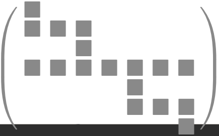
Exploiting hierarchical structure

Order rows and columns of Γ according to the location of the maxima of $\psi_{j,k}$. Γ has sparsity structure

$$(3,0)\ (2,0)\ (3,1)\ (1,0)\ (3,2)\ (2,1)\ (3,3)$$

$\Gamma =$  ,

$\Gamma = SS'$ where

$S =$  .

Recursive sparsity pattern

$$S^1 = (s_{11})$$

$$S^J = \begin{bmatrix} S_l^{J-1} & 0 & 0 \\ S_{cl} & s_{cc} & S_{cr} \\ 0 & 0 & S_r^{J-1} \end{bmatrix} \left.\begin{matrix} 2^{J-1}-1 \\ 1 \\ 2^{J-1}-1 \end{matrix}\right\}$$

Hierarchical back-substitution

A hierarchical back-substitution problem of the form

$$\underbrace{\begin{bmatrix} S_l & 0 & 0 \\ S_{cl} & s_{cc} & S_{cr} \\ 0 & 0 & S_r \end{bmatrix}}_{(m+1+m)\times(m+1+m)} \begin{bmatrix} X_l \\ x_c \\ X_r \end{bmatrix} = \begin{bmatrix} B_l \\ b_c \\ B_r \end{bmatrix}$$

can be recursively solved by solving the back-substitution problems $S_l X_l = B_l$, $S_r X_r = B_r$ and setting

$$x_c = s_{cc}^{-1} \cdot (b_c - S_{cl} X_l - S_{cr} X_r)$$

Factorization in quasi linear time

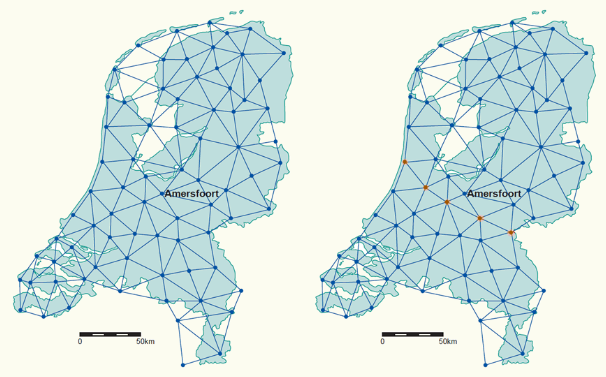
$$\begin{bmatrix} A_l & A'_{cl} & 0 \\ A_{cl} & a_{cc} & A_{cr} \\ 0 & A'_{cr} & A_r \end{bmatrix} = \begin{bmatrix} S_l & 0 & 0 \\ S_{cl} & s_{cc} & S_{cr} \\ 0 & 0 & S_r \end{bmatrix} \begin{bmatrix} S_l & S_{cr} & 0 \\ 0 & s_{cc} & 0 \\ 0 & S_{cr} & S_r \end{bmatrix}$$

$$= \begin{bmatrix} S_l S'_l & S'_l S_{cl} & 0 \\ S'_{cl} S_l & s_{cc}^2 + S_{cl} S'_{cl} + S_{cr} S'_{cr} & S'_r S_{cr} \\ 0 & S'_{cr} S_r & S_r S'_r \end{bmatrix}$$

Here $A_l = S_l S'_l$ and $A_r = S_r S'_r$ are two hierarchical factorization problems of level $J-1$, $A_l = S'_{cl} S_l$ and $A_r = S'_{cr} S_r$ are hierarchical back-substitution problems and

$$s_{cc} = \sqrt{a_{cc} - S_{cl} S'_{cl} + S_{cr} S'_{cr}}.$$

Approximative sparse inversion using nested dissection



[2]

Application: Nonparametric inference for diffusion process

$$\mathrm{d}X_t = b_0(X_t)\,\mathrm{d}t + \mathrm{d}W_t \tag{1}$$

Prior $P(J \geq j) \geq C\exp(-2^j)$ and

$$b = \sum_{j=1}^{J} \sum_{k=0}^{2^{j-1}-1} Z_{j,k}\psi_{j,k}$$

$$M\Xi^J \geq_{pd} \Gamma^J \geq_{pd} m\Xi^J$$

where $\alpha = \frac{1}{2}$, $\Xi^J = \mathrm{diagm}(2^{-2(j-1)\alpha}, 1 \leq j \leq J, 0 \leq k < 2^{j-1})$

Gaussian inverse problem

Likelihood

$$p(X \mid b) = \exp\Big(\int_0^T b(X_t)\,\mathrm{d}X_t - \frac{1}{2}\int_0^T b^2(X_t)\,\mathrm{d}t\Big)$$

$$\mu_\iota^J = \int_0^T \psi_\iota(X_t)\,\mathrm{d}X_t, \quad \iota = 1, \dots, 2^J - 1$$

$$G_{\iota,\iota'}^J = \int_0^T \psi_\iota(X_t)\psi_{\iota'}(X_t)\,\mathrm{d}t, \quad \iota, \iota' = 1, \dots, 2^J - 1.$$

Γ^J and G^J have the same sparsity pattern

Conjugate posterior

For fix level J,

$$Z^J \mid J, X \sim \mathcal{N}(\Sigma^J \mu^J, \Sigma^J)$$

where $\Sigma^J = (\Gamma^J + G^J)^{-1}$.
On J a reversible jump algorithm can be used.

Posterior contraction rates (periodic case)

Besov norm, supremum norm for $f = \sum \sum z_{j,k} \psi_{j,k}$

$$\|f\|_\alpha = \sup_{j \geq 1, k} 2^{(j-1)\alpha} |z_{j,k}| \quad \|f\|_\infty \leq \sum_j \max_k |z_{j,k}|$$

Sieves

$$B_{L,M} = \left\{ \sum_{j=1}^{L} \sum_{k=0}^{2^{j-1}-1} z_{j,k} \psi_{j,k} \colon 2^{\alpha(j-1)} |z_{j,k}| \leq M, j, k = \dots \right\}$$

Rate

$$T^{-\frac{\beta}{1+2\beta}} \log(T)^{\frac{\beta}{1+2\beta}} \quad \beta \geq \alpha$$

Anderson's lemma

If $X \sim N(0, \Sigma_X)$ and $Y \sim N(0, \Sigma_Y)$ independent with $\Sigma_X \leq_{pd} \Sigma_Y$ positive definite, then then for all symmetric convex sets $P(Y \in C) \leq P(X \in C)$.

Summary

- Midpoint displacement construction of Gauss-Markov processes
- Corresponding Gaussian Markov random field
- Chordal graphs and perfect elimination orderings
- Sparse Cholesky decomposition
- Rates for randomly truncated prior

Image sources

[1] http://math.stackexchange.com/questions/251856/area-enclosed-by-2-dimensional-random-curve
[2] http://kartoweb.itc.nl/geometrics/reference%20surfaces/body.htm